Preface

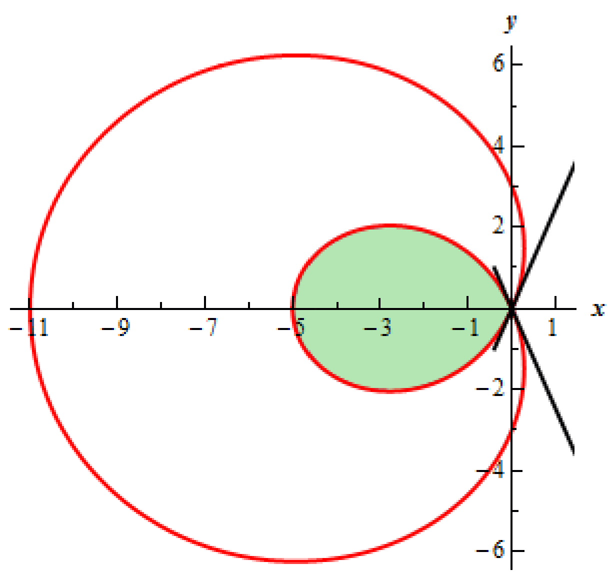
Here are the solutions to the practice problems for my Calculus II notes. Some solutions will have more or less detail than other solutions. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you've reached the level of working the harder problems then you will probably already understand the basics fairly well and won't need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven't been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

Area with Polar Coordinates

1. Find the area inside the inner loop of $r = 3 - 8\cos\theta$.

Step 1
First, here is a quick sketch of the graph of the region we are interested in.



Step 2

Now, we'll need to determine the values of θ that the graph goes through the origin (indicated by the black lines on the graph in the previous step).

There are easy enough to find. Because they are where the graph goes through the origin we know that we must have $r = 0$. So,

$$3 - 8\cos\theta = 0$$
$$\cos\theta = \tfrac{3}{8} \qquad \Rightarrow \qquad \theta = \cos^{-1}\left(\tfrac{3}{8}\right) = 1.1864$$

This is the angle in the first quadrant where the graph goes through the origin.

We next need the angle in the fourth quadrant. We need to be a little careful with this second angle. We need to always remember that the limits on the integral we'll eventually be computing must go from smaller to larger value. Also, as the angle moves from the smaller to larger value they must trace out the boundary curve of the region we are interested in.

From a quick sketch of a unit circle we can quickly get two possible values for the angle in the fourth quadrant.

$$\theta = 2\pi - 1.1864 = 5.0968 \qquad\qquad \theta = -1.1864$$

Depending upon the problem we are being asked to do either of these could be the one we need. However, in this case we can see that if we use the first angle (*i.e.* the positive angle) we actually end up tracing out the outer portion of the curve and that isn't what we want here. However, if we use the second angle (*i.e.* the negative angle) we will trace out the inner loop as we move from this angle to the angle in the first quadrant.

So, for this particular problem, we need to use $\theta = -1.1864$.

The ranges of θ for this problem is then $-1.1864 \le \theta \le 1.1864$.

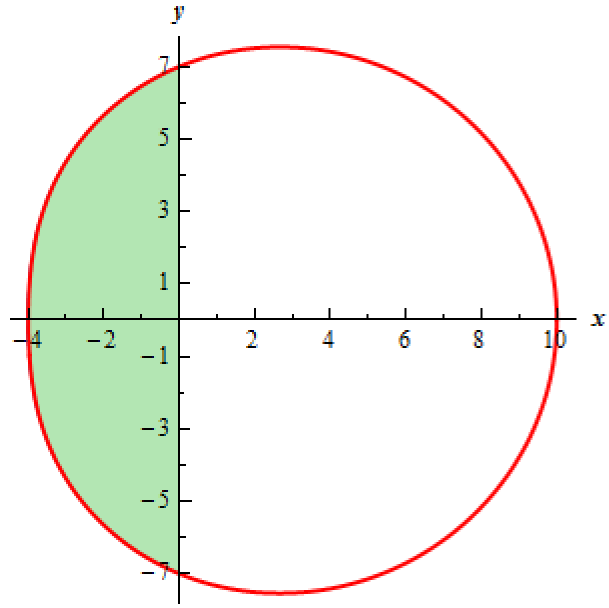
Step 3
The area of the inner loop is then,

$$\begin{aligned} A &= \int_{-1.1864}^{1.1864} \tfrac{1}{2}\left(3 - 8\cos\theta\right)^2 d\theta \\ &= \tfrac{1}{2}\int_{-1.1864}^{1.1864} 9 - 48\cos\theta + 64\cos^2\left(\theta\right) d\theta \\ &= \tfrac{1}{2}\int_{-1.1864}^{1.1864} 9 - 48\cos\theta + 32\left(1 + \cos\left(2\theta\right)\right) d\theta \\ &= \tfrac{1}{2}\int_{-1.1864}^{1.1864} 41 - 48\cos\theta + 32\cos\left(2\theta\right) d\theta \\ &= \tfrac{1}{2}\left(41\theta - 48\sin\left(\theta\right) + 16\sin\left(2\theta\right)\right)\Big|_{-1.1864}^{1.1864} = \boxed{15.2695} \end{aligned}$$

Make sure you can do the trig manipulations required to do these integrals. Most of the integrals in this section will involve this kind of manipulation. If you don't recall how to do them go back and take a look at the Integrals Involving Trig Functions section.

2. Find the area inside the graph of $r = 7 + 3\cos\theta$ and to the left of the *y*-axis.

Step 1
First, here is a quick sketch of the graph of the region we are interested in.



Step 2
For this problem there isn't too much difficulty in getting the limits for the problem. We will need to use the limits $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ to trace out the portion of the graph to the left of the *y*-axis.

Remember that it is important to trace out the portion of the curve defining the area we are interested in as the θ's increase from the smaller to larger value.

Step 3
The area is then,

$$\begin{aligned} A &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \tfrac{1}{2}\left(7 + 3\cos\theta\right)^2 d\theta \\ &= \tfrac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 49 + 42\cos\theta + 9\cos^2(\theta)\,d\theta \\ &= \tfrac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 49 + 42\cos\theta + \tfrac{9}{2}\left(1 + \cos(2\theta)\right)d\theta \\ &= \tfrac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \tfrac{107}{2} + 42\cos\theta + \tfrac{9}{2}\cos(2\theta)\,d\theta \\ &= \tfrac{1}{2}\left(\tfrac{107}{2}\theta + 42\sin(\theta) + \tfrac{9}{4}\sin(2\theta)\right)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \boxed{42.0376} \end{aligned}$$

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Make sure you can do the trig manipulations required to do these integrals. Most of the integrals in this section will involve this kind of manipulation. If you don't recall how to do them go back and take a look at the Integrals Involving Trig Functions section.

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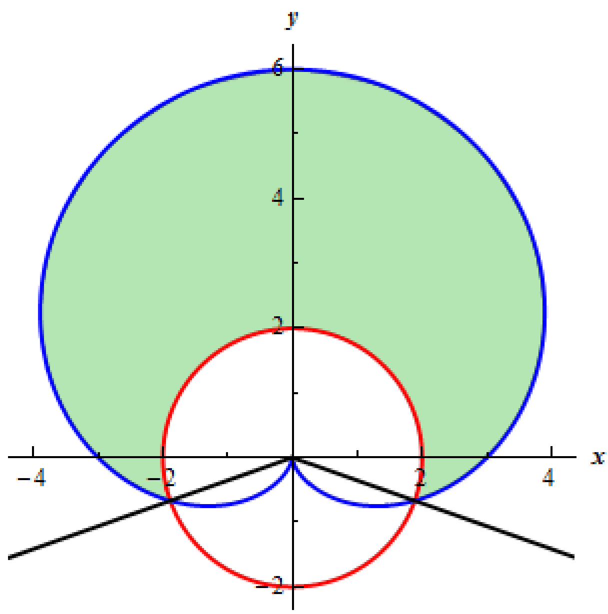
3. Find the area that is inside $r = 3 + 3\sin\theta$ and outside $r = 2$.

Step 1
First, here is a quick sketch of the graph of the region we are interested in.



Step 2
Now, we'll need to determine the values of θ where the graphs intersect (indicated by the black lines on the graph in the previous step).

There are easy enough to find. Because they are where the graphs intersect we know they must have the same value of *r*. So,

$$3 + 3\sin\theta = 2$$
$$\sin\theta = -\tfrac{1}{3} \qquad \Rightarrow \qquad \theta = \sin^{-1}\left(-\tfrac{1}{3}\right) = -0.3398$$

This is the angle in the fourth quadrant where the graphs intersect.

From a quick sketch of a unit circle we can quickly get the angle in the third quadrant where the two graphs intersect.

$$\theta = \pi + 0.3398 = 3.4814$$

The ranges of θ for this problem is then $-0.3398 \le \theta \le 3.4814$.

Step 3
From the graph we can see that $r = 3 + 3\sin\theta$ is the "outer" graph for this region and $r = 2$ is the "inner" graph.

The area then,

$$\begin{aligned} A &= \int_{-0.3398}^{3.4814} \tfrac{1}{2}\left[\left(3+3\sin\theta\right)^2 - \left(2\right)^2\right] d\theta \\ &= \tfrac{1}{2}\int_{-0.3398}^{3.4814} 5 + 18\sin\theta + 9\sin^2\left(\theta\right) d\theta \\ &= \tfrac{1}{2}\int_{-0.3398}^{3.4814} 5 + 18\sin\theta + \tfrac{9}{2}\left(1-\cos\left(2\theta\right)\right) d\theta \\ &= \tfrac{1}{2}\int_{-0.3398}^{3.4814} \tfrac{19}{2} + 18\sin\theta - \tfrac{9}{2}\cos\left(2\theta\right) d\theta \\ &= \tfrac{1}{2}\left(\tfrac{19}{2}\theta - 18\cos\left(\theta\right) - \tfrac{9}{4}\sin\left(2\theta\right)\right)\Big|_{-0.3398}^{3.4814} = \boxed{33.7074} \end{aligned}$$

Make sure you can do the trig manipulations required to do these integrals. Most of the integrals in this section will involve this kind of manipulation. If you don't recall how to do them go back and take a look at the Integrals Involving Trig Functions section.

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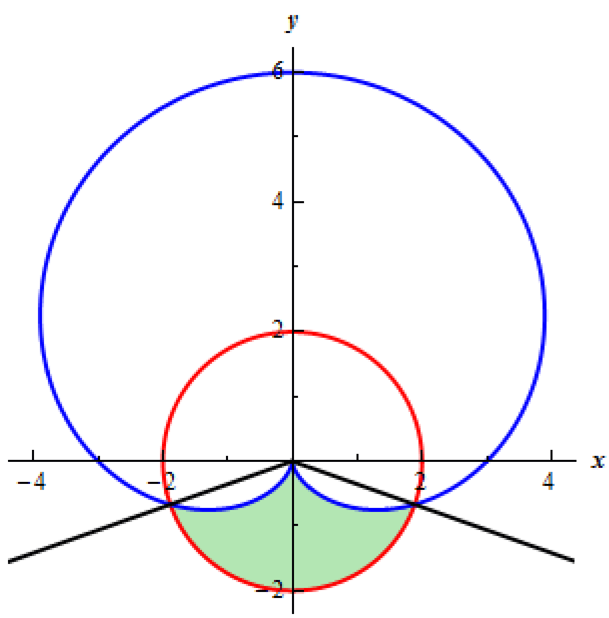
4. Find the area that is inside $r = 3 + 3\sin\theta$ and outside $r = 2$.

Step 1
First, here is a quick sketch of the graph of the region we are interested in.



Step 2
Now, we'll need to determine the values of θ where the graphs intersect (indicated by the black lines on the graph in the previous step).

There are easy enough to find. Because they are where the graphs intersect we know they must have the same value of *r*. So,

$$
\begin{aligned}
3+3\sin\theta &= 2 \\
\sin\theta &= -\tfrac{1}{3} \qquad \Rightarrow \qquad \theta = \sin^{-1}\left(-\tfrac{1}{3}\right) = -0.3398
\end{aligned}
$$

This is one possible value for the angle in the fourth quadrant where the graphs intersect.

From a quick sketch of a unit circle we can quickly get the angle in the third quadrant where the two graphs intersect.

$$\theta = \pi + 0.3398 = 3.4814$$

Now, we'll have a problem if we use these two angles for our area integral. Recall that the angles must go from smaller to larger values and as they do that they must trace out the boundary curves of the enclosed area. These two clearly will not do that. In fact they trace out the area from the previous problem.

To fix this problem it is probably easiest to use a quick sketch of a unit circle to see that another value for the angle in the fourth quadrant is,

$$\theta = 2\pi - 0.3398 = 5.9434$$

Using this angle along with the angle we already have in the third quadrant will trace out the area we are interested in.

Therefore, the ranges of θ for this problem is then $3.4814 \le \theta \le 5.9434$.

Step 3
From the graph we can see that $r = 2$ is the "outer" graph for this region and $r = 3+3\sin\theta$ is the "inner" graph.

The area then,

$$\begin{aligned} A &= \int_{3.4814}^{5.9434} \tfrac{1}{2}\left[(2)^2 - (3+3\sin\theta)^2\right] d\theta \\ &= \tfrac{1}{2}\int_{3.4814}^{5.9434} -5 - 18\sin\theta - 9\sin^2(\theta)\, d\theta \\ &= \tfrac{1}{2}\int_{3.4814}^{5.9434} -5 - 18\sin\theta - \tfrac{9}{2}\left(1-\cos(2\theta)\right) d\theta \\ &= \tfrac{1}{2}\int_{3.4814}^{5.9434} -\tfrac{19}{2} - 18\sin\theta + \tfrac{9}{2}\cos(2\theta)\, d\theta \\ &= \tfrac{1}{2}\left(-\tfrac{19}{2}\theta + 18\cos(\theta) + \tfrac{9}{4}\sin(2\theta)\right)\Big|_{3.4814}^{5.9434} = \boxed{3.8622} \end{aligned}$$

Do not get too excited about all the minus signs in the second step above. Just because all the terms have minus signs in front of them does not mean that we should get a negative value from out integral!

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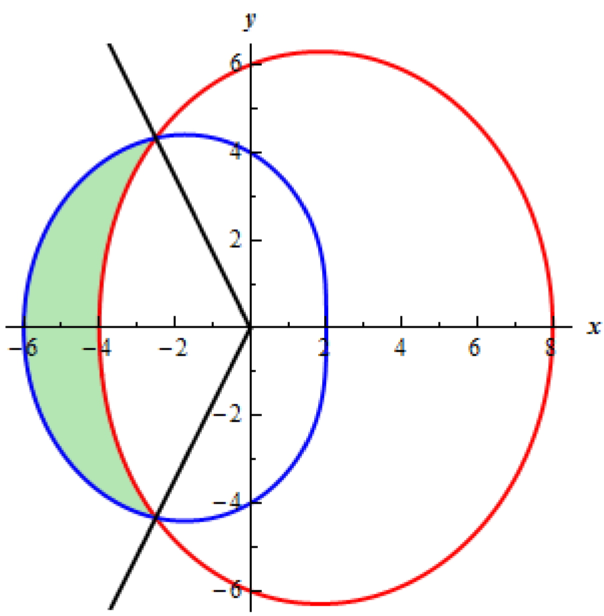
5. Find the area that is inside $r = 4 - 2\cos\theta$ and outside $r = 6 + 2\cos\theta$.

Step 1
First, here is a quick sketch of the graph of the region we are interested in.



Step 2
Now, we'll need to determine the values of θ where the graphs intersect (indicated by the black lines on the graph in the previous step).

There are easy enough to find. Because they are where the graphs intersect we know they must have the same value of r. So,

$$6+2\cos\theta = 4-2\cos\theta$$
$$\cos\theta = -\tfrac{1}{2} \qquad \Rightarrow \qquad \theta = \cos^{-1}\left(-\tfrac{1}{2}\right) = \tfrac{2\pi}{3}$$

This is the value for the angle in the second quadrant where the graphs intersect.

From a quick sketch of a unit circle we can quickly see two possible values for the angle in the third quadrant where the two graphs intersect.

$$\theta = 2\pi - \tfrac{2\pi}{3} = \tfrac{4\pi}{3} \qquad\qquad \theta = -\tfrac{2\pi}{3}$$

Now, we need to recall that the angles must go from smaller to larger values and as they do that they must trace out the boundary curves of the enclosed area. Keeping this in mind and we can see that we'll need to use the positive angle for this problem. If we used the negative angle we'd be tracing out the "right" portions of our curves and we need to trace out the "left" portions of our curves.

Therefore, the ranges of θ for this problem is then $\tfrac{2\pi}{3} \le \theta \le \tfrac{4\pi}{3}$.

Step 3
From the graph we can see that $r = 4-2\cos\theta$ is the "outer" graph for this region and $r = 6+2\cos\theta$ is the "inner" graph.

The area then,

$$\begin{aligned} A &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \tfrac{1}{2}\left[\left(4-2\cos\theta\right)^2 - \left(6+2\cos\theta\right)^2\right] d\theta \\ &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} -10-20\cos\theta \, d\theta \\ &= \left(-10\theta - 20\sin\left(\theta\right)\right)\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} = \boxed{13.6971} \end{aligned}$$

Do not get too excited about all the minus signs in the integral. Just because all the terms have minus signs in front of them does not mean that we should get a negative value from out integral!

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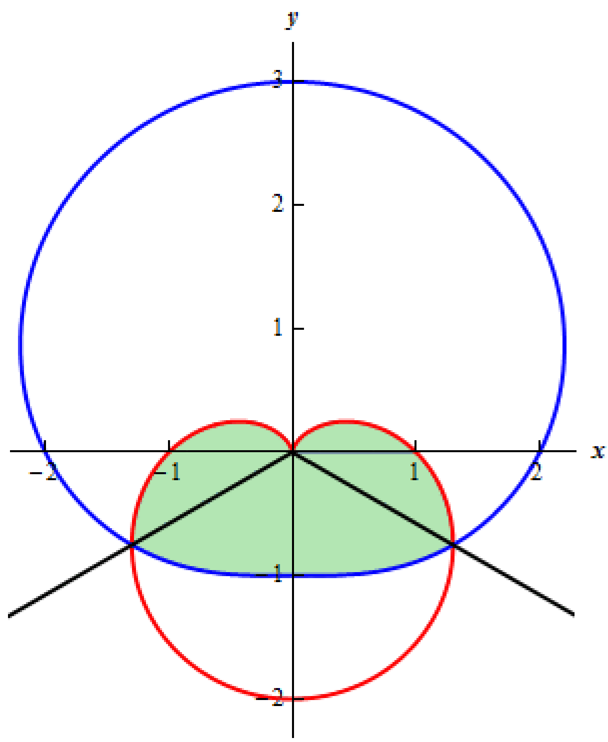
6. Find the area that is inside both $r = 1-\sin\theta$ and $r = 2+\sin\theta$.

Step 1
First, here is a quick sketch of the graph of the region we are interested in.



Step 2
Now, we'll need to determine the values of θ where the graphs intersect (indicated by the black lines on the graph in the previous step).

There are easy enough to find. Because they are where the graphs intersect we know they must have the same value of *r*. So,

$$2+\sin\theta=1-\sin\theta$$
$$\sin\theta=-\tfrac{1}{2} \qquad \Rightarrow \qquad \theta=\sin^{-1}\left(-\tfrac{1}{2}\right)=-\tfrac{\pi}{6}$$

This is one possible value for the angle in the fourth quadrant where the graphs intersect. From a quick sketch of a unit circle we can see that a second possible value for this angle is,

$$\theta=2\pi-\tfrac{\pi}{6}=\tfrac{11\pi}{6}$$

From a quick sketch of a unit circle we can quickly get a value for the angle in the third quadrant where the two graphs intersect.

$$\theta=\pi+\tfrac{\pi}{6}=\tfrac{7\pi}{6}$$

Okay. We now have a real problem. Recall that the angles must go from smaller to larger values and as they do that they must trace out the boundary curves of the enclosed area.

If we use $-\tfrac{\pi}{6}\le\theta\le\tfrac{7\pi}{6}$ we actually end up tracing out the large "open" or unshaded region that lies above the shaded region.

Likewise, if we use $\frac{7\pi}{6} \le \theta \le \frac{11\pi}{6}$ we actually end up tracing out the smaller "open" or unshaded region that lies below the shaded region.

In other words, we can't find the shaded area simply by using the formula from this section.

Step 3
As we saw in the previous step we can't just compute an integral in order to get the area of the shaded region. However, that doesn't mean that we can't find the area of the shaded region. We just need to work a little harder at it for this problem.

To find the area of the shaded area we can notice that the shaded area is really nothing more than the remainder of the area inside $r = 2 + \sin\theta$ once we take out the portion that is also outside $r = 1 - \sin\theta$.

Another way to look at is that the shaded area is simply the remainder of the area inside $r = 1 - \sin\theta$ once we take out the portion that is also outside $r = 2 + \sin\theta$.

We can use either of these ideas to find the area of the shaded region. We'll use the first one for no other reason that it was the first one listed.

If we knew the total area that is inside $r = 2 + \sin\theta$ (which we can find with a simple integral) and if we also knew the area that is inside $r = 2 + \sin\theta$ and outside $r = 1 - \sin\theta$ then the shaded area is nothing more than the difference between these two areas.

Step 4
Okay, let's start this off by getting the total area that is inside $r = 2 + \sin\theta$. This can be found by evaluating the following integral.

$$\begin{aligned} A &= \int_0^{2\pi} \tfrac{1}{2}\left(2+\sin\theta\right)^2 d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} 4 + 4\sin\theta + \sin^2(\theta)\, d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} 4 + 4\sin\theta + \tfrac{1}{2}\left(1-\cos(2\theta)\right) d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} \tfrac{9}{2} + 4\sin\theta - \tfrac{1}{2}\cos(2\theta)\, d\theta \\ &= \tfrac{1}{2}\left(\tfrac{9}{2}\theta - 4\cos(\theta) - \tfrac{1}{4}\sin(2\theta)\right)\Big|_0^{2\pi} = \boxed{\tfrac{9\pi}{2}} \end{aligned}$$

Note that we need to do a full "revolution" to get all the area inside $r = 2 + \sin\theta$ and so we used the range $0 \le \theta \le 2\pi$ for this integral.

Step 5
Now, the area that is inside $r = 2 + \sin\theta$ and outside $r = 1 - \sin\theta$ is,

$$
\begin{aligned}
A &= \int_{-\frac{\pi}{6}}^{\frac{11\pi}{6}} \tfrac{1}{2}\left[\left(2+\sin\theta\right)^2 - \left(1-\sin\theta\right)^2\right] d\theta \\
&= \tfrac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{11\pi}{6}} 3+6\sin\theta \, d\theta \\
&= \tfrac{1}{2}\left(3\theta - 6\cos\left(\theta\right)\right)\Big|_{-\frac{\pi}{6}}^{\frac{11\pi}{6}} = \boxed{3\pi}
\end{aligned}
$$

Step 6
Finally the shaded area is simply,

$$
A = \tfrac{9\pi}{2} - 3\pi = \boxed{\tfrac{3\pi}{2}}
$$
